Lesson Outline

1. Fixed Point Method
   Fixed Point Iteration

2. Rate of Convergence
Fixed Point Iteration

If the equation, $f(x) = 0$ is rearranged in the form

$$x = g(x)$$

then an iterative method may be written as

$$x_{n+1} = g(x_n) \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (1)

where $n$ is the number of iterative steps and $x_0$ is the initial guess. This method is called the **Fixed Point Iteration** or **Successive Substitution Method**.
Definition of Fixed Point

If \( c = g(c) \), the we say \( c \) is a **fixed point** for the function \( g(x) \).

Theorem

**Fixed Point Theorem (FPT)**

Let \( g \in C[a, b] \) be such that \( g(x) \in [a, b] \), for all \( x \) in \([a, b]\). Suppose, in addition, that \( g'(x) \) exists on \((a, b)\). Assume that a constant \( K \) exists with

\[
|g'(x)| \leq K < 1, \quad \text{for all } x \text{ in } (a, b)
\]

Assume that \( c \) in \((a, b)\) is a fixed point for \( g \). Then if \( x_0 \) is any point in \((a, b)\), the sequence

\[
x_{n+1} = g(x_n) \quad n = 0, 1, 2, \ldots
\]

converges to the unique fixed point \( c \). *(Proof - B&F page 59)*
Example: Given $f(x) = x^3 - 7x + 2 = 0$ in $[0,1]$. Find a sequence that $\{x_n\}$ that converges to the root of $f(x) = 0$ in $[0,1]$.

Answer: Rewrite $f(x) = 0$ as $x = \frac{1}{7}(x^3 + 2)$. Then $g(x) = \frac{1}{7}(x^3 + 2)$ and $g'(x) = \frac{3x^2}{7} < \frac{3}{7}$ for all $x \in [0,1]$. Hence, by the FPT the sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{1}{7}(x^3 + 2)$$

converges to a root of $x^3 - 7x + 2 = 0$
Example: Solve $f(x) = x^3 - x - 1 = 0$ on $(1, 2)$.

Answer: Note $f(-1) = -1$ and $f(2) = 5$, \therefore by the IVT a root exists on $(1,2)$. Set $g(x) = (1 + x)^{\frac{1}{3}}$. Note that $g'(x) = \frac{1}{3}(1 + x)^{-2/3}$. So, on $(1,2)$ we have

$$\frac{1}{3(1 + 2)^{2/3}} < g'(x) < \frac{1}{3(1 + 1)^{2/3}}$$

\therefore $0 < g'(x) < \frac{1}{3(2^{2/3})} = K$

and $|g'(x)| \leq K < 1$ on $(1,2)$. By the FPT the sequence

$$x_{n+1} = (1 + x_n)^{\frac{1}{3}}$$

will converge to a fixed point on $(1,2)$. 
\[ x_{n+1} = (1 + x_n)^{\frac{1}{3}}, \quad x_0 = 1.3 \]

- \( x_0 = 1.3 \)
- \( x_1 = 1.320006122 \)
- \( x_2 = 1.323822354 \)
- \( x_3 = 1.324547818 \)
- \( x_4 = 1.324685639 \)
- \[ \vdots \]
- \( x_{11} = 1.324717957 \)
- \( x_{12} = 1.324717957 \)
- \( x_{13} = 1.324717957 \)

Example 2 & 3 of B & F. (Page 57-58)
Definition

Suppose that \( \{x_n\} \) is a sequence of numbers generated by an algorithm, and the limit of the sequence is \( s \). If

\[
\lim_{n \to \infty} \frac{|x_{n+1} - s|}{|x_n - s|^p} = K, \quad K \neq 0
\]

for some positive constants \( K \) and \( p \), then we say that the sequence \( \{x_n\} \) converges to \( s \) with \( p \) being the order of convergence.

If \( p = 1 \), convergence is linear.
If \( p = 2 \), convergence is quadratic.
Larger values of \( p \) imply faster rates of convergence.