Roots of Equations - The Bisection Method

M311 - Chapter 2

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Lesson Outline

1 Bisection Method
   Intermediate Value Theorem
   Bisection Method Algorithm
   Bisection Error Analysis
Intermediate Value Theorem (IVT)

If a function \( f(x) \) is continuous on \([a, b]\) and \( K \) is a number between \( f(a) \) and \( f(b) \), then there exist a number \( c \) in \((a, b)\) for which \( f(c) = K \).

Refer to Page 9 - Burden & Faires for sample diagram

If a function \( f(a) \) and \( f(b) \) have opposite signs
i.e if \( f(a)f(b) < 0 \), by the IVT
\[ \exists \ a \ number \ c \ in \ (a, b) \ for \ which \ f(c) = 0. \]
Examples of IVT

1. Does the function $x^5 - 2x^3 + 3x^2 - 1 = 0$ have a solution in [0,1]? - refer to Page 9 of B & F for solution.

2. Does $h(x) = x \sin x - 1$ have a solution in [0, 2].
   Answer: Compute $h(0)$ and $h(1)$.
   
   $h(0) = -1.000000$ and $h(2) = 0.818595$

   Since $h(0)h(1) < 0$ there is a root in (0,2)
Bisection Method Algorithm to find root in \([a, b]\)

1. Bisect \([a, b]\) into two halves \([a, c]\) and \([c, b]\) where \(c = \frac{a + b}{2}\).
2. Identify the interval containing the root by checking the signs of \(f(a)f(c)\) and \(f(c)f(b)\).
3. If \(f(a)f(c) < 0\) then interval \([a, c]\) has the root. Otherwise the other interval \([c, b]\) has the root.
4. Bisect the new interval that contains the root and repeat steps 1-3.
5. At each step take the midpoint of the interval as the most updated approximation of the root.
6. Stop the procedure after a specified number of iterations or when the width of the interval containing the root is less than a given tolerance \(\varepsilon\).
Example 1. The root of $e^x - 2 = 0$ is known to exist in $[0,2]$. Use 8 iterations to find an approximate value of the root (or find an approximate value of the root to within a tolerance of $\varepsilon$)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{iter. \#} & a & c & b & f(a) & f(c) & f(b) \\
\hline
1 & 0.0000 & 1.0000000 & 2.000000 & -1.0000 & 0.7183 & 5.3891 \\
2 & 0.0000 & 0.5000000 & 1.000000 & -1.0000 & -0.3513 & 0.7183 \\
3 & 0.5000 & 0.7500000 & 1.000000 & -0.3513 & 0.1170 & 0.7183 \\
4 & 0.5000 & 0.6250000 & 0.750000 & -0.3513 & -0.1318 & 0.1170 \\
5 & 0.6250 & 0.6875000 & 0.750000 & -0.1318 & -0.0113 & 0.1170 \\
6 & 0.6875 & 0.7187500 & 0.750000 & -0.0113 & 0.0519 & 0.1170 \\
7 & 0.6875 & 0.7031250 & 0.718750 & -0.0113 & 0.0201 & 0.0519 \\
8 & 0.6875 & 0.6953125 & 0.703125 & -0.0113 & 0.0043 & 0.0201 \\
\hline
\end{array}
\]
Bisection Error Analysis

Bisection Method Theorem

If the bisection algorithm is applied to a continuous function \( f \) on an interval \([a, b]\), where \( f(a)f(b) < 0 \), then, after \( n \) steps, an approximate root will have been computed with error at most

\[
\frac{b - a}{2^{n+1}}
\]

For Proof, refer to handout.
Bisection Method

Intermediate Value Theorem

Bisection Method Algorithm

Bisection Error Analysis

\[ \frac{b_0 - a_0}{2} \]

\[ |r - c_0| \]

\[ a_0 \quad r \quad c_0 \quad b_0 \]
If an error tolerance has been prescribed in advance, it is possible to determine the number of steps required in the bisection method. Suppose that we want $|r - c_n| < \epsilon$. Then it is necessary to solve the following inequality for $n$:

$$\frac{b - a}{2^{n+1}} < \epsilon$$

By taking logarithms, we obtain

$$n > \frac{\log(b - a) - \log(2\epsilon)}{\log 2}$$
Example 2. How many steps of the bisection algorithm are needed to compute the root of a function $f(x)$ to a precision of $\varepsilon = 0.01$ on the interval $[0, 2]$?

**Answer.** $a = 0$ and $b = 2$.

\[
\frac{b - a}{2^{n+1}} < \varepsilon \\
\frac{2 - 0}{2^{n+1}} < 0.01 \\
2^n > 100 \\
n > \frac{\log 100}{\log 2} = 6.64
\]

Thus no more than $n = 7$ iterations would be needed to achieve the convergence to within 0.01.